RELATIVE SHEAR DEFORMATION OF NON-NEWTONIAN LIQUIDS IN IMPELLER INDUCED FLOW

Kamil WICHTERLE[1,]* and Pavel MITSCHKA

Institute of Chemical Process Fundamentals, Academy of Sciences of the Czech Republic, 165 02 Prague 6-Suchdol, Czech Republic; e-mail: [1] kamil.wichterle@vsb.cz

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Flow induced by rotating bodies is studied from the viewpoint of the shear of liquid particles. By solution of the boundary layer equations for Newtonian and power-law non-Newtonian liquids at a rotating disc, a typical velocity field was obtained. The shear distribution of particles leaving the disc edge shows that the most important deformation occurs in the boundary layer. Distribution of shear from the viewpoint of the volume flow rate is also presented. Application of the results to the prediction of particle-breakup dynamics at rotating impellers is discussed.
Key words: Boundary layer; Rotating disc; Shear; Particle breakup; Non-Newtonian liquids.

Rotating boundary layers at mixing agitators or centrifugal pump rotors are spaces where high-level shear stress takes place. This is important from the viewpoint of micromixing, if striae of different composition enter the boundary layer. Deformable particles (gas bubbles, drops of immiscible liquid, elastic particles, biological bodies and various agglomerates) may change reversibly or irreversibly their shape or can be broken to smaller elements. The deformation can also intensify mass transfer. Both the level of shear stress and duration of its action on a given particle is important for prediction of the process result.

Theories of micromixing and interaction of liquid with small particles were mostly developed from a phenomenological concept of turbulence[1,2]. Processes in high-shear-rate laminar boundary layers have not been studied extensively. In our experimental study[3], we discovered that the shear rate at turbine impeller blades had similar properties as that one in laminar boundary layer at a rotating disc. The same conclusion was obtained by measuring shear rates at the centrifugal pump impeller[4] until crossing the turbulence threshold at Reynolds number $Re \approx 2 \cdot 10^5$. Application of the laminar boundary layer approach was found fruitful even as a model for the prediction of terminal drop size in agitated vessels[5]. Knowledge of the total relative shear deformation

* Present address: VSB-Technical University Ostrava, 708 33 Ostrava-Poruba, Czech Republic.

imposed on a particle in a given streamline is an additional information which can serve for understanding the dynamics of the drop breakup.

In the present paper, the shear distribution in a liquid pumped by a rotating disc is calculated from the velocity field, obtained by solving the pertinent boundary layer equations.

THEORETICAL

At higher Reynolds numbers, the flow around a finite rotating disc (Fig. 1) with radius R can be, in the zone r < R, approximated by the solution of flow around an infinite rotating disc. Simple boundary conditions for the latter case enable transformation of complete Navier–Stokes equations for Newtonian liquids, and the boundary layer equations (high-Reynolds-number approximation) for non-Newtonian liquids into a set of ordinary differential equations. Their solution can be generally described for $r < R$ by the formulas:

$$v_r = \omega r\, F(\zeta) \ , \tag{1}$$

$$v_\varphi = \omega r\, G(\zeta) \ , \tag{2}$$

$$v_z = \omega r \left(\frac{r^2 \omega^{2-n} \rho}{K}\right)^{-1/(1+n)} H(\zeta) \ , \tag{3}$$

where the dimensionless coordinate is defined as

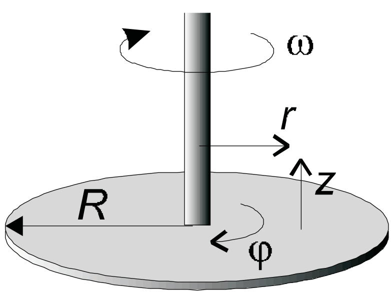

FIG. 1
Coordinate system for the rotating disc flow

$$\zeta \equiv \left(\frac{z}{r}\right)\left(\frac{r^2\omega^{2-n}\rho}{K}\right)^{1/(1+n)} \quad . \tag{4}$$

Special functions $F(\zeta)$, $G(\zeta)$, $H(\zeta)$ can be obtained by solution of the set of ordinary differential equations formulated for Newtonian liquids ($n = 1$) by Karman[6]. These functions, first computed approximately by Cochran[7], are presented as tables in the Schlichting monograph[8]. Today they can be determined numerically with any accuracy requested. The problem for non-Newtonian liquids was analyzed using a somewhat different terminology by Mitschka[9–11], who computed the velocity profiles for power-law liquids with flow indexes $0.1 \le n \le 1.5$.

Any liquid element in the space $r < R$ approaches the disc, where its rotation is accelerated, and it is moved radially by the action of centrifugal forces. We assume that after leaving the disc region, local velocity differences vanish, and additional deformation outside the disc region can be neglected. Therefore, the shear deformation in the range $r < R$ is in the focus of our interest.

The local shear rate, γ, can be calculated using two major components of the kinematics tensor, $\partial v_r/\partial z$, $\partial v_\varphi/\partial z$. After introduction of the Reynolds number,

$$Re = \frac{R^2\omega^{2-n}\rho}{K} \quad , \tag{5}$$

it can be expressed by the dimensionless function

$$\frac{\gamma}{\omega} = \left(\frac{r}{R}\right)^{2/(1+n)} Re^{1/(1+n)} \left(F'(\zeta)^2 + G'(\zeta)^2\right)^{1/2} \quad . \tag{6}$$

By symbols F' and G', derivatives of functions F and G of the argument ζ are shortly denoted.

Total relative shear deformation, D, of a fluid particle is given by the product of local shear rate and time element integrated along the streamline coming from infinity and leaving the disc edge $r = R$ at the distance z_0,

$$D = -\int_{z_0}^{\infty} \gamma \frac{\mathrm{d}t}{\mathrm{d}z}\,\mathrm{d}z \quad . \tag{7}$$

As the axial velocity is $v_z \equiv \mathrm{d}z/\mathrm{d}t$, we can apply relations (*3*), (*4*) and (*6*) in the integrand (*7*), which gives

$$D(\zeta_0,n) = Re^{1/(1+n)} \int_{\zeta_0}^{\infty} \left(\frac{r}{R}\right)^{2/(1+n)} \frac{(F'^2 + G'^2)^{1/2}}{-H}\,\mathrm{d}\zeta \quad , \tag{8}$$

where

$$\zeta_0 \equiv \left(\frac{z_0}{R}\right) Re^{1/(1+n)} \quad . \tag{9}$$

As a streamline projected on the plane φ = const is

$$\frac{\mathrm{d}r}{\mathrm{d}\zeta} = r\,\frac{F}{H} \quad , \tag{10}$$

we obtain by integration

$$\ln\frac{r}{R} = \int_{\zeta_0}^{\zeta} \frac{F}{H}\,\mathrm{d}\zeta \quad . \tag{11}$$

When a new special function $B(\zeta_0,n)$ is introduced using known functions F, G, H as

$$B(\zeta_0,n) = \int_{\zeta_0}^{\infty} \exp\left(\frac{2}{1+n}\int_{\zeta_0}^{\zeta}\frac{F}{H}\,\mathrm{d}\zeta\right)\frac{(F'^2+G'^2)^{1/2}}{-H}\mathrm{d}\zeta \quad , \tag{12}$$

it is evident that B has the meaning of a normalized relative shear deformation and relation (*9*) can be written simply as

$$D = Re^{1/(1+n)}B(\zeta_0,n). \tag{13}$$

For the particular case of Newtonian liquids (n = 1) where

$$F = \frac{-1}{2}H' \quad , \tag{14}$$

we obtain simply

$$B(\zeta_0,1) \equiv (-H(\zeta_0))^{1/2}\int_{\zeta_0}^{\infty}\frac{(F'^2+G'^2)^{1/2}}{(-H)^{3/2}}\,\mathrm{d}\zeta \quad . \tag{15}$$

The normalized relative shear deformation $B(\zeta_0,n)$ for selected values of n is plotted in Fig. 2. There are apparently two different regions, which is usual for the problems where the boundary layer approach can be applied. The most significant deformation occurs for $\zeta_0 < 3$, which is apparent from Fig. 3, where the product $\zeta_0 B(\zeta_0,n)$ is presented. For non-Newtonian liquids, the transition from the boundary layer (low ζ_0) to the bulk (high ζ_0) is considerably smoother than for $n = 1$.

The volume of liquid passing the disc in a given distance above it is given by function $F(\zeta_0,n)$. Therefore the plot of F *vs* B in Fig. 4 is just the distribution of the normalized shear in the liquid leaving the disc. The main part of the pumped volume has a value B close to unity both for Newtonian and non-Newtonian liquids.

Cumulative volume rate of the flow which leaves the disc through the layer $0 < z < z_0$ on one side of the disc is $Q(z_0)$, and can be calculated as

$$Q = 2\pi\int_0^{z_0} rv_r \, \mathrm{d}z = 2\pi\omega R^3 Re^{-1/(1+n)} M(\zeta_0,n) \ . \qquad (16)$$

The continuity equation gives

$$F = -\frac{1}{2}H' - \frac{(1-n)}{2(1+n)}F'\frac{\zeta}{2} \ , \qquad (17)$$

and then the normalized flow rate, $M(\zeta_0,n)$ defined by the relation

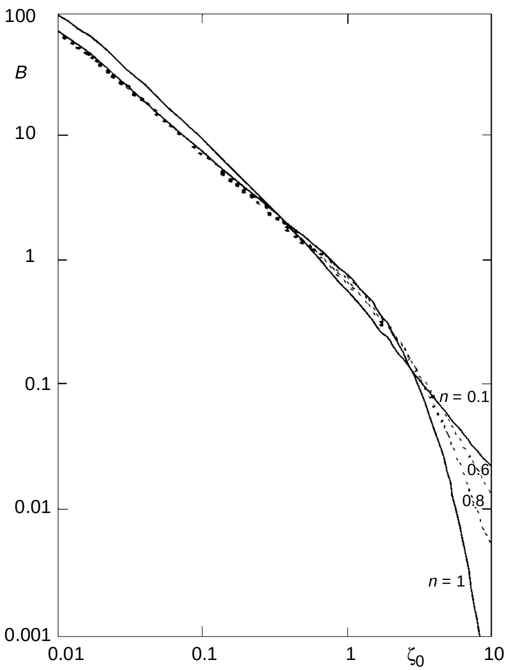

FIG. 2
Normalized relative shear deformation B for the rotating disc flow

$$M(\zeta_0,n) = \int_0^{\zeta_0} F(\zeta,n)\, \mathrm{d}\zeta \qquad (18)$$

can also be expressed directly using functions *H* and *F*:

$$M(\zeta_0,n) = -\frac{n+1}{3n+1} H(\zeta_0,n) + \frac{1-n}{3n+1}\, \zeta_0\, F(\zeta_0,n) \ . \qquad (19)$$

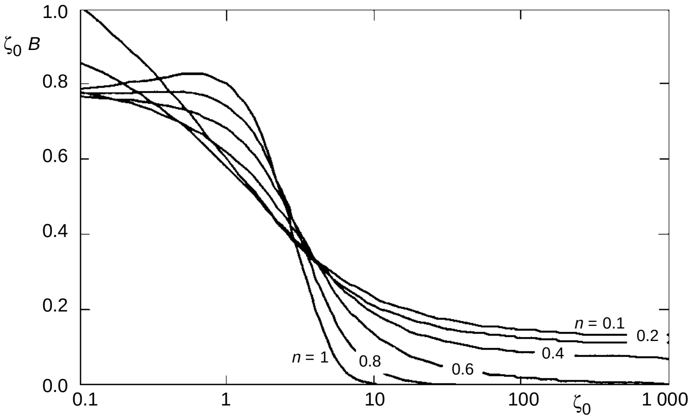

FIG. 3
Product $\zeta_0\, B(\zeta_0,n)$ as a function of ζ_0 for selected values of *n* for the rotating disc flow

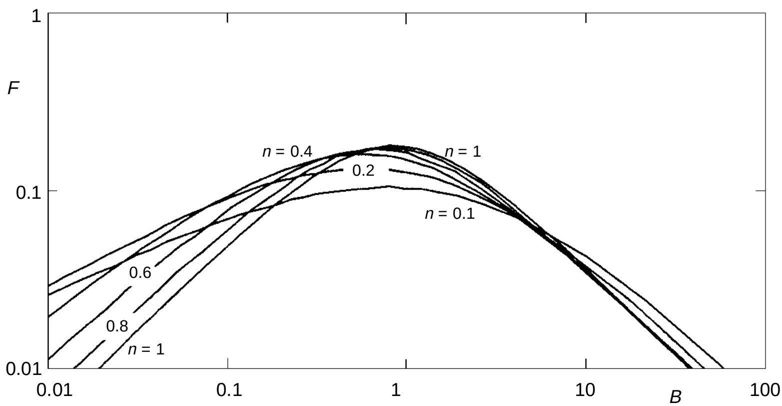

FIG. 4
Normalized relative shear deformation distribution in liquid passing the disc

As it is seen from Fig. 5, the cumulative volume flow rate through a narrow layer adjacent to a disc increases approximately with the square of distance. In this range it is more suitable to read the values of the normalized volume flow rate from the plot of a function $M(\zeta_0,n)/\zeta_0^2$, presented in Fig. 6.

The dependence of M on B shown in Fig. 7 indicates what volume flow rate, $Q = 2\pi\omega R^3\, Re^{-1/(1+n)}M$, of pumped liquid is subjected to total relative shear deformations higher than $D = Re^{1/(1+n)}B$. There exists an asymptote, $\lim_{M\to\infty}(MB^2) = f(n)$, and therefore a better reading for the more important higher deformation range is provided by the plot in Fig. 8.

DISCUSSION

Deformation of fluid continuum particles itself can be used to determine the striation thickness distribution and its time changes in an equipment with a rotating disc or a similar impeller. When an elementary volume with lamellar[12] (striation) thickness L traverses the boundary layer, the striation thickness assumes a new value which is approximately $L/(1 + D)$ for small D, or L/D for larger D.

Another important quantity, which can be estimated with a knowledge of relative shear deformation, is the size of deformable particles (bubbles, immiscible drops, breakable plastic particles, or assemblages of smaller objects) dispersed in the liquid. Evolution of their size as an effect of the passage along a rotor generally follows the striation thickness. However, there are several additional factors:

First, there is usually a certain limit of breakup, given by a terminal particle size, d_P. For drops and bubbles, *e.g.*, it is controlled by interface forces. For agglomerates, cohesion forces of various nature operate.

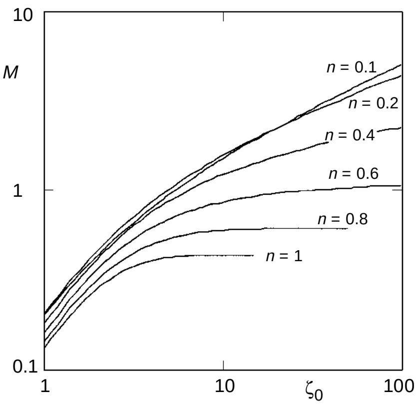

FIG. 5

Cumulative volume flow rate M as a function of ζ_0 for selected values of n for the rotating

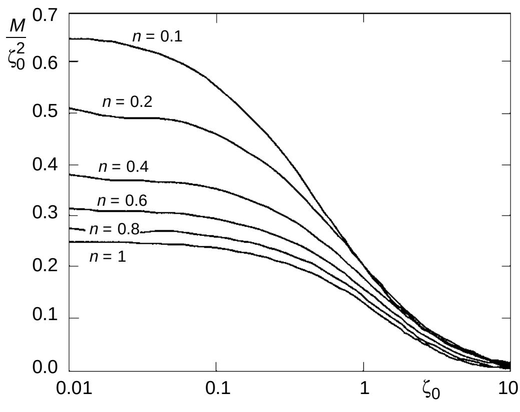

FIG. 6

Cumulative flow rate M for lower range of ζ_0

Second, while bubbles and low-viscosity particles are deformed similarly to the ambient liquid, relative shear deformation of viscous drops, D_D, can be estimated by the relation $D_D = (\mu_C/\mu_D)\, D$, where μ_C/μ_D is the viscosity ratio of the continuous and dispersed phases. There is a quantitative difference between low-viscosity and high-viscosity drops. To be broken, the low-viscosity drop needs to cross the condition $D >> 1$, which means that $B >> 0.1$ is satisfactory even at $Re_M = 100$. Practically all the liquid pumped by the rotating disc is subjected to an essentially higher relative shear deformation and, as shown earlier[5], in common mixing equipment, the mixing time necessary

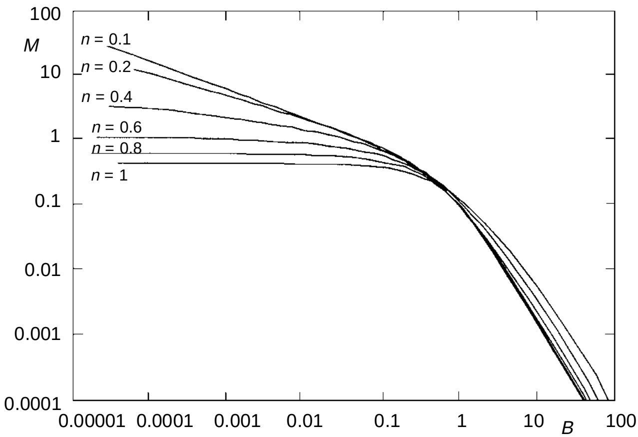

FIG. 7
Normalized volume flow rate, *M*, of liquid subjected to relative shear deformations higher than $Re^{1/(1+n)}B$

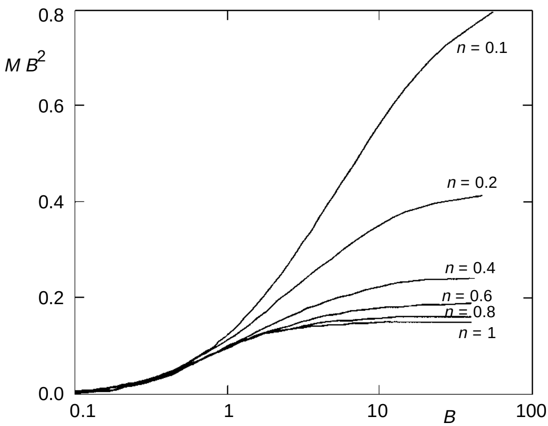

FIG. 8
Normalized cumulative volume flow rate, *M*, of liquid subjected to relative shear deformations higher than $Re^{1/(1+n)}B$ for the high-deformation region

for passing all the liquid volume along the impeller is $t_M \approx 30\ Re_M^{1/2}/N$. In small pilot-plant units, t_M usually makes several minutes; on industrial scale, however, it may be several hours. For high-viscosity drops, high values of $B > 1$ are required, and the volume flow rate decreases according to the relation $M \approx 0.15/B^2$ (as apparent from Fig. 8). The related time to pass the breakup region can be estimated as $t_M \approx 65(\mu_D/\mu_C)^2/(NRe_M^{1/2})$, which indicates a considerable slow-down of the process at high-viscosity drops. Estimation of the drop breakup rate based on the deformation knowledge is in accordance with common experience while the theories of turbulence have no explanation for the long-time response.

Third, elasticity and plasticity of the particles causes that, after a small relative shear deformation of the ambient liquid, the particles relax again, and there is no response to their size. There is some critical total relative shear deformation of the ambient liquid, necessary for the breakup which is at least $D = 2$ for low-viscosity objects and considerably higher for other particles. For example, from the data by Converti[13], we have estimated the critical total deformation $D = 3\ 000$ of a liquid substrate, which destroys the studied class of cells in a bioreactor, and the assumption of $M \approx 0.15/B^2$ gives an acceptable model for the process dynamics.

CONCLUSIONS

The velocity field at an infinite high-speed rotating disc was calculated by solving the boundary layer equations. The equations can be transformed to ordinary differential equations and easily solved for Newtonian liquids and, neglecting some, probably minor terms, this can be also done for the non-Newtonian case. It is known that this solution describes well the velocity field near a rotating impeller. By integration of the local shear stress along streamlines coming from the bulk towards the impeller and then to the impeller tip, the total relative shear deformation was computed. It can be seen that there are no essential differences in the dimensionless deformation distribution for non-Newtonian and Newtonian liquids. If some difference were pointed out, it would be the fast decay of velocity behind the Newtonian boundary layer while a measurable induced flow is apparent beyond the non-Newtonian boundary layer.

Total relative shear deformation, D, for streamlines entering the boundary layer is proportional to $Re^{1/(1+n)}$ and decreases first inversely proportionally to the distance from the impeller surface. Later on and outside the boundary layer, the total deformation decreases faster.

Distribution of D with respect to the cumulative volume flow rate shows that prevailing volume pumped by the impeller is subjected to the total shear deformation of the order $D = Re^{1/(1+n)}B$.

A possibility of applying the knowledge of total relative shear deformation to predict micromixing and the particle-breakup dynamics at impellers was shown.

APPENDIX

While radius, R, and angular velocity, ω, is used to characterize the length and speed in fluid mechanics papers, in chemical engineering correlations, the diameter, $d = 2R$, and revolutions per second, $N = \omega/2\pi$, are preferred instead. Velocity components are then

$$v_r = \pi N d F(\zeta) \quad , \tag{1a}$$

etc. Other formulas could be transformed by using

$$Re_{\mathrm{M}} \equiv N^{2-n} d^2 \rho / K = \pi^{2-n}/2^n \, Re \tag{5a}$$

$$\zeta \equiv (2\pi^{2-n})^{1/(1+n)} z/d Re_{\mathrm{M}}^{1/(1+n)} \tag{4a}$$

$$D = (\pi^{2-n}/2^n)^{1/(1+n)} \, Re_{\mathrm{M}}^{1/(1+n)} \, B(\zeta) \tag{13a}$$

$$Q/(Nd^3) = (\pi^{3n}/2)^{1/(1+n)} \, Re_{\mathrm{M}}^{-1/(1+n)} \, M(\zeta) \quad . \tag{16a}$$

For Newtonian liquids, it is particularly:

$$v_r = 3.14 Nd \, F(\zeta) \quad , \tag{1b}$$

$$Re_{\mathrm{M}} \equiv Nd^2 \, \rho/\mu = 1.57 Re \tag{5b}$$

$$\zeta = 2.51 z/d \, Re_{\mathrm{M}}^{1/2} = 2.51 z \, (N\rho/\mu)^{1/2} \quad . \tag{4b}$$

$$D = 1.25 Re_{\mathrm{M}}^{1/2} \, B(\zeta) \tag{13b}$$

$$Q/(Nd^3) = 3.94 Re_{\mathrm{M}}^{-1/2} \, M(\zeta) \quad . \tag{16b}$$

SYMBOLS

B	normalized total relative shear deformation
d	impeller diameter, m
D	total relative shear deformation
F	dimensionless radial velocity
G	dimensionless angular velocity
H	dimensionless axial velocity
K	consistency coefficient, Pa s^n
M	dimensionless cumulative volume flow
N	rotation speed, s^{-1}
n	power-law flow index
Q	cumulative volume flow rate, $m^3\,s^{-1}$
r	radial coordinate, m
R	impeller radius, m
Re	Reynolds number
t_M	mixing time, s
v	velocity, m s^{-1}
z	axial coordinate, m
γ	shear rate, s^{-1}
μ	viscosity, Pa s
ρ	density, kg m^{-3}
ζ	dimensionless axial coordinate
ω	angular speed, s^{-1}

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